4-may Lecture 22

Monday, May 4, 2026 8:26 AM

last time saw $R(s) = \frac{P(s)}{Q(s)}$ where degree $P(s) <$ degree $Q(s)$

relates to partial fraction decomposition

Rule 1. Linear Factor Partial Fractions

decomposition of $R(s)$ corresponds to linear factor $s-a$ with multiplicity, k is a sum of partial fractions of form:

$R(s)$ RHS $\quad \frac{A}{s-a} + \frac{B}{(s-a)^2} + \cdots$ where A, B are constants

$\frac{P(s)}{Q(s)}$

relates back to $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

decompose $R(s)$ into partial fraction form to ease Laplace transform application

$\mathcal{L}\{f(t)\} = F(s)$

$\mathcal{L}^{-1}\{F(s)\} = f(t)$

ex. to apply an inverse Laplace transform:

given $F(s) = \frac{3}{2s-4} = \frac{3}{2(s-2)} = \frac{3}{2} \cdot \frac{1}{s-2}$ $\quad a = 2$

$= \boxed{\frac{3}{2} e^{2t}}$

ex. given $F(s) = \frac{1}{s^2+4s+4}$

$= \frac{1}{(s+2)^2}$ no partial fraction nec.

$= \frac{1}{(s-(-2))^2}$ $\quad 1!$ $\quad a = -2$

$n+1 \Rightarrow n = 1$

$\mathcal{L}\{e^{at} t^n\} = \frac{n!}{(s-a)^{n+1}}$

$f(t) = e^{-2t} \cdot t^1$

$\boxed{f(t) = te^{-2t}}$

$$\boxed{f(t) = te^{-2t}}$$

ex. given $F(s) = \frac{5-2s}{s^2+7s+10}$

$$= \frac{5-2s}{(s+2)(s+5)} = \frac{A}{s+2} + \frac{B}{s+5}$$

$$5-2s = As + 5A + Bs + 2B$$

$$5 = 5A + 2B$$

$$-5\,(-2 = A + B)$$

$$5 = 5A + 2B$$
$$10 = -5A - 5B$$
$$15 = -3B \Rightarrow B = -5$$
$$A = 3$$

$$F(s) = 3 \cdot \frac{1}{s+2} - 5 \cdot \frac{1}{s+5}$$

$$\boxed{x(t) = 3e^{-2t} - 5e^{-5t}}$$

Rule 2 Quadratic Factor Partial Fractions

use when have factors that are irreducible beyond $(s-a)^2 + b^2$

RHS form: $\frac{As+B}{(s-a)^2+b^2} + \ldots$

recall: $X(s) = \frac{3s+19}{s^2+6s+34} = \frac{3s+19}{(s+3)^2 + 25}$ → 5^2

last time, completed the square

use $s+3$, 5 in numerator

$$\frac{3s+19}{(s+3)^2+5^2} = A \cdot \frac{s+3}{(s+3)^2+5^2} + B \cdot \frac{5}{(s+3)^2+5^2}$$

$$3s+19 = As + 3A + 5B$$

$A = 3$
$B = 2$

$$X(s) = 3 \cdot \frac{s+3}{(s+3)^2+5^2} + 2 \cdot \frac{5}{(s+3)^2+5^2}$$

$a = -3$

$$\mathcal{L}\{e^{at}\cos kt\} = \frac{s-a}{(s-a)^2+k^2}$$

$$X(s) = 3 \cdot \frac{s+3}{(s+3)^2+5^2} + 2 \cdot \frac{5}{(s+3)^2+5^2}$$

$k=5$

$$\boxed{x(t) = 3e^{-3t}\cos 5t + 2e^{-3t}\sin 5t}$$

$\mathcal{L}\{e^{at}\cos kt\} = \frac{s-a}{(s-a)^2+k^2}$

$\mathcal{L}\{e^{at}\sin kt\} = \frac{k}{(s-a)^2+k^2}$

revisit 7.1

$$\mathcal{L}\{3t\} = 3\,\mathcal{L}\{t\}$$
$$= 3 \cdot \frac{1}{s^2} = \boxed{\frac{3}{s^2}}$$

recall:

$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

$\Gamma(n+1) = n\Gamma(n) = n!$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$

$\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$

k

$$\mathcal{L}\{\sin 2t + \cos 2t\}$$
$$= \boxed{\frac{2}{s^2+4} + \frac{s}{s^2+4}}$$

n

$$\mathcal{L}\{\sqrt{t}\} = \mathcal{L}\{t^{1/2}\} = \frac{\left(\frac{1}{2}\right)!}{s^{3/2}}$$
$$= \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{3/2}} = \frac{1\sqrt{\pi}}{2s^{3/2}}$$

$$\boxed{\mathcal{L}\{\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}}}$$

$$\mathcal{L}\{t^{5/2}\} = \frac{\frac{5}{2}!}{s^{7/2}}$$
$$= \boxed{\frac{15\sqrt{\pi}}{8\,s^{7/2}}}$$

$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ $\quad n! = n\Gamma(n)$

$n+1$

$$\frac{5}{2}! = \frac{5}{2}\Gamma\left(\frac{5}{2}\right)$$
$$= \frac{5}{2} \cdot \frac{3}{2}\Gamma\left(\frac{3}{2}\right)$$
$$= \frac{15}{4} \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$
$$= \frac{15}{8}\sqrt{\pi}$$

inverse Laplace transform review:

$\mathcal{L}\{t^n\} \Leftarrow \frac{n!}{s^{n+1}}$

ex. given $F(s) = \frac{3}{s^4} = 3 \cdot \frac{1}{s^4}$ $\leftarrow n=3$

$$= 3 \cdot \frac{1}{6}t^3$$
$$= \boxed{\frac{1}{2}t^3}$$

since $\mathcal{L}\{t^3\} = \frac{3!}{s^4}$

$$= \frac{6}{s^4}$$

$\mathcal{L}^{-1}: \frac{1}{\ldots} = \frac{1}{\ldots} \cdot \frac{6}{\ldots}$

$= \boxed{\frac{1}{2} t^3}$

$= \frac{\ }{s^4}$

then $\frac{1}{s^4} = \frac{1}{6} \cdot \frac{6}{s^4}$

$= \frac{1}{6} \underbrace{\mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\}}_{t^3}$

Section 7.2 review

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

helpful to convert:

$$\mathcal{L}\{x'\} = sX(s) - x(0)$$

$$\mathcal{L}\{x''\} = s^2X(s) - sx(0) - x'(0)$$

and ofc $\mathcal{L}\{x\} = X(s)$

ex. solve $x'' - x' - 2x = 0$ given $x(0) = 0$ $x'(0) = 2$

$$s^2X(s) - s\cancel{x(0)}^{\,0} - \cancel{x'(0)}^{\,2} - (sX(s) - \cancel{x(0)}^{\,0}) - 2X(s) = 0$$

$$s^2 \underline{X(s)} \quad - 2 \quad - s\underline{X(s)} \quad - 2\underline{X(s)} = 0$$

$$+2 \qquad\qquad +2$$

$(s^2 - s - 2)X(s) = 2$ solve for $X(s)$

$$X(s) = \frac{2}{s^2 - s - 2} = \frac{2}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1} \qquad A = 2/3 \quad B = -2/3$$

$$= \frac{2}{3}\left(\frac{1}{s-2} - \frac{1}{s+1}\right)$$

$$\boxed{x(t) = \frac{2}{3}\left(e^{2t} - e^{-t}\right)}$$